FACILITATING EFFECTIVE LEARNING IN ABSTRACT ALGEBRA: A STUDY OF TEACHING PRACTICES AND PROOF COMPREHENSION

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Introduction

Abstract algebra occupies a central place in the undergraduate mathematics curriculum, and yet it is often perceived as one of the most difficult topics to learn and to teach. Its formalism, level of abstraction, and departure from earlier mathematical experiences can leave students disoriented, unsure how to approach new concepts or how to connect them to what they already know.

To explore how these issues are experienced and addressed in practice, we conducted interviews with six lecturers teaching level 2, 3 and 4 abstract algebra course units at the University of Manchester. For ease of reference, we will refer to these lecturers as interviewees throughout. Their reflections revealed two insights. First, there is a unanimous recognition that students often struggle with understanding what the structures mean and they mentioned a problem with students' ability to understand proofs and consequently reproduce them or construct their own proofs. Second, many interviewees intuitively employed strategies that align with pedagogical research, but were often unaware of the research supporting these decisions.

This raises some broader pedagogical questions: what kind of understanding is being built when a student learns about algebraic structures, not just how to manipulate them, but what they mean? How does that understanding differ from the knowledge an expert draws on? What makes abstract ideas so resistant to understanding in the first place? In what follows, we'll introduce some theoretical frameworks for answering these questions, and then provide scholarly-informed activities to help students go from simply doing maths, to understanding it.

1. General Theory

To understand the difficulties students face in abstract algebra, we first need to outline key learning theories. This section introduces schemas, novice vs expert learning and other principles that shape how students form and refine mathematical understanding.

1.1. **The nature of difficulty in abstract algebra.** To abstract is to recognise what is shared across different experiences. It's how we go from feeling "sweet" to forming the concept of sweetness, or from noticing patterns in movement to the idea of symmetry. Abstraction is a kind of lasting mental image: you wouldn't

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confuse a table for a chair, even though you can sit on the table. In education, researchers often distinguish between abstraction as an activity and abstraction as a product: the resulting concept. These ideas form hierarchies: concepts derived from sensory experience are called primary, and those built from other concepts are called secondary. If concept A is derived from concept B, then A is a higher-order abstract concept (see [43]). In mathematics, especially in fields like abstract algebra, we ascend even further: factor structures involve sets of sets, and algebraic structures can interact and give rise to higher-order maps. This conceptual leap, while powerful, presents deep challenges for both learners and teachers.

A key challenge in learning abstract algebra is the gap between procedural fluency and conceptual understanding. Skemp (1976) [42] highlights the difference between knowing how to apply and manipulate techniques (procedural or instrumental understanding) and having a deeper understanding of structures (conceptual or relational understanding). In abstract algebra, the latter is arguably more important for learning. An example to support that is given by the extensive research done on issues of undergraduate students being able to calculate cosets and factor structures but failing to answer questions testing their understanding of them [5].

1.2. **What does learning mean?** There are many different definitions of learning, but in this project we are focusing on the one provided by Kirschner, Clark and Sweller: "learning is a change in long-term memory" [26]. Long-term memory represents what we know about the world and informs how we act. It is believed to have no known limits (see eg. [31]), and the knowledge it holds is outside of our awareness, often dormant, until it's activated and enters our working memory. Working memory is where our thoughts occur in the here and now. Extensive studies by Sweller(1998) [44] show that working memory has finite capacity, and if it is overloaded, then learning cannot occur.

Long-term memory holds schemas. A schema is a structured cluster of related knowledge that allows us to treat multiple elements of information as a single unit. In other words, schemas are the mental frameworks that give meaning to new information and allow us to process it more efficiently. They are built through the process described in the first paragraph: connecting new ideas to what is already known, and they make learning possible [43].

There are two pathways that represent how learning occurs. Firstly, a student takes ideas from long-term memory, the existing knowledge and information, and makes connections with the newly learned information in working memory. As Willingham (2009) [52] writes, "understanding is remembering in disguise." Almost everything we learn depends on something else already. But in order for accurate learning to occur, the student needs to be able to relate the correct old ideas.

In the opposite direction, when a student retrieves knowledge from long-term memory into working memory, its schema is "expanding". This plays on the integrative function of a schema. Think of the schema as a map; the more you explore, the more comprehensive your map will be. If you visit a place often, you'll more naturally remember the layout and your favourite places. All in all, schemas are stored in long-term memory and they act as a tool for future learning. From now on, we will focus on abstract conceptual schemas. More on schema theory in Piaget (1977)[37] and Piaget (1928) [38]. One reason students' schemas remain fragile is the mismatch between their concept image and formal definitions: this is the focus of section 2.

1.3. **Novice vs Expert learning.** Students cannot build richer schemas without aligning their informal understandings with formal definitions, a difficulty that often arises in examples, proofs, and abstraction more broadly. This distinction becomes clear when comparing novices to experts' schemas integrate formal definitions and flexible representations, novices often rely heavily on incomplete or informal understandings. This difference extends beyond definitions: novices and experts approach learning with fundamentally different strategies and levels of analysis (this claim uses Hill and Schneider (2006) [21]). In his 2017 work, Didau [15] recognises two hallmarks of expertise: automaticity of foundational knowledge, and ability to see deep structure within domains of expertise.

These differences are especially stark when it comes to proof comprehension. Research supports this: Selden and Selden (2003) [41] observed that students tend to concentrate heavily on the superficial aspects of supposed proofs. Later, Inglis and Alcock (2012) [24] supported this finding through eye-tracking studies, showing that undergraduate students, unlike expert mathematicians, spent more time focusing on the algebraic sections of proofs and paid less attention to the accompanying text where key logical statements are typically clarified.

Studying the approaches of experts and novices when tackling physics problems, Chi et. al. (1981) [11] concluded that while experts tend to abstract the physical principles of the problem and solve the problem representation, novices tend to base their strategy on the specific particularities of the problems. This illustrates the latter hallmark that Didau recognised. To further support this insight in abstract algebra, an interviewee reported: "It seems like some students can't make the connections between related concepts. They think of these examples as different concepts, instead of all together. And despite how many times you tell them 'it's the same thing, it's the same thing', some people don't [understand]." On a similar note, another interviewee noted that in the tutorial sessions for the introductory level 2 group theory course, a student struggled to see how they could apply the Fundamental Isomorphism Theorem in a different setting, with a different notation.

1.4. **Cognitive Load Theory.** As established, working memory is limited. Cognitive Load Theory (eg. Sweller (1988) [44]) is used in practice to inform educational strategies and design instructions that avoid the overload of working memory. When it becomes overloaded, students are more likely to misunderstand the material being taught to them. Simply put, if they get overwhelmed with poorly organised materials, a poorly lit lecture theatre etc., learning is disrupted. As we have seen, for Kirschner (and for us), learning is a change in long-term memory, and if the pathway from working memory is blocked, then no change happens, therefore learning cannot occur.

We will put in more formal terms the ideas expressed above. Cognitive psychologists recognise three main types of cognitive load:

1. **Intrinsic load**. Intrinsic load relates to the inherent difficulty of the subject matter being learnt, and is determined by the complexity of the material and the prior knowledge of the learner [6, p. 150]. Coming back to the discussion around higher-order abstract concepts, abstract algebra is rich in cognitively demanding ideas and intricate structures. These hierarchies of abstraction make abstract algebra concepts and proofs resistant to learning. To address this complexity, interviewees emphasised the importance of building a strong foundation before introducing higher-order abstractions. Several of them reflected on how crucial it is to help students retrieve and connect prior knowledge so that new ideas do not feel overwhelming. We will return to these reflections, and explore practical strategies for achieving this, in section 3.

2. **Extraneous load**. While intrinsic load is, to some extent, unavoidable, extraneous load is detrimental for learning but preventable. It takes the form of badly designed instructions and the environment that the material is presented in. As we have just listed, it is could be badly organised materials, incomplete explanations and sometimes even bad handwriting. In short, it is extraneous because it is load that otherwise would be used for learning.

3. **Germane load**. This often refers to the process of thinking involved in learning. It is the pathway of refining and developing one's schemas, and ideally, instructions should be focused in maximising this.

Cognitive Load Theory is an intuitive idea, not meant to make thinking easier, but to facilitate the process of learning. You can read more about this in [4].

2. From Theory to Practice

The following section will first highlight the main theoretical framework regarding the common shortcomings for learners. Then, we will present research-informed strategies which could make understanding concepts in abstract algebra easier.

2.1. Concept Images and Example Spaces.

Tall and Vinner (1981) [45, p. 152] introduced the term "concept images" to mean "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." In contrast with concept definitions (what we might ordinarily call a mathematical definition), concept images are usually the first thing that students consult when asked a technical problem (Vinner (1991)[46]). Concept images are different from schemas, as the former are only concerned with one object or process. Following Feudel and Unger (2024) [17], we take "understanding a concept" to mean having a rich concept image.

This naturally leads to the question: how can such rich concept images be fostered? Watson and Mason (2006) [48, p. 4] did extensive work on the idea of example spaces, which they define as the collection of examples and boundary examples a learner has access to that fulfill particular pedagogical functions. These allow learners "to see the general through the particular, to generalise, to experience the particular as exemplary, to appreciate a technical term, theorem, proof, or proof structure, and so on."

Crucially, a concept image can also diverge from the formal definition. Alcock and Simpson(2010) [1, p.18-19], for example, documented students' difficulties with cosets: many could manipulate symbols procedurally but lacked a sense of what a coset means, often confusing objects and processes. Lecturers in our study observed similar issues. One interviewee teaching level 4 abstract algebra commented: "When I think of a Lie Algebra I have a stack of friendly examples that I know are representative. But students don't have that." This remark highlights the need for activities that deliberately broaden students' example spaces and refine their concept images.

However, as Tall and Vinner noted, concept images can also contain elements that conflict with the formal definition. When these conflicts go unaddressed, especially in the early knowledge acquisition phase, they can develop into robust misconceptions that are difficult to unlearn. Understanding how these misconceptions arise, and how to address them, is therefore essential for supporting students' conceptual understanding and will make up the bulk of this section.

2.2. **Misconceptions.**

2.2.1. *Why are misconceptions formed?* As Kirschner et. al. (2006) note, learning can be understood as "a change in long-term memory." The challenge, however, lies in the fact that students do not enter lectures as blank slates awaiting the acquisition of accurate knowledge. Instead, they arrive with a complex array of prior experiences and preconceptions, some of which are inaccurate. These existing cognitive frameworks can give rise to misconceptions that impede the assimilation of new concepts.

We mentioned above the early knowledge acquisition phase of learning, simply meaning the first encounter of a student to a concept or skill. In this time, as we've seen, novices hold naturally more fragmented and incomplete concept images and if they don't have access to a comprehensive collection of examples, they risk developing inaccurate concept images that are harder to correct due to a conflict factor [45]. To understand more in depth how misconceptions in abstract algebra are formed, we will explore these cognitive factors.

Traditionally, in the early knowledge acquisition phase lecturers start with the definition and a couple of examples. This can potentially generate two issues. The first issue is that the internalisation of a definition comes with a preconceived concept image. The language used in a definition can evoke an inaccurate concept image. Dubinsky et. al. [16] did extensive work on the cognitive processes of learners in abstract algebra, and found that the terms "set" and "group" are treated similarly for some time by novices, therefore their concept images exclude the group operation. Therefore, pre-existing knowledge can clash with the internalisation of a new concept.

The second issue is that students tend to focus on superficial properties when engaging with a new concept. Studies by Hazzan (1999) [19] and Melhuish et. al. (2020) [34] observed that students tend to emphasize features such as the sign of elements when considering ring operations — viewing zero as the identity element for addition and using the usual addition and multiplication operations on the rational numbers. This superficial focus, combined with limited exposure to diverse examples, can reinforce misconceptions.

Interviewees leading tutorials in the level 2 “Groups and Geometry” course underlined the common mistakes around Lagrange’s theorem, one of the most important theorems in introductory group theory. This corroborates the existing studies that support this, see Hazzan and Leron (1996) [20]. We divided the reported issues into two categories. The first one is related to proof comprehension, which is something that will be explained and addressed more in depth in section 3. The second one relates to precisely this erroneous understanding of concepts involved. Most interviewees realised that students struggle when cosets are introduced, which supports the theory that higher-order abstraction is harder to comprehend. Another thing noted is the misuse of the “naive converse” of Lagrange’s theorem, namely, the incorrect assumption that if the order of a subset divides the order of the group, then the subset must be a subgroup.

To reiterate the above, if students are not given access to a rich example space, that is when misconceptions are likely to arise. To address this, the next subsection will explore the role of variation theory, which highlights how carefully structured variation in examples can support more accurate concept formation.

2.2.2. *Variation theory.* According to Marton and Pang (2013) [30], the thesis of variation theory is “meanings are acquired from experiencing differences against a background of sameness, rather than from experiencing sameness against a background of difference.”

In his 1986 study [53], Wilson points out that learners can be distracted by irrelevant properties of an example and can’t tell where the boundaries of a definition are. The presence of examples and non-examples can provide a better picture in what is, and what is not included in a definition.

An interviewee teaching a level 3 course reports his observations on how students struggle with the boundaries of a definition: “When I think of a Principal Ideal Domain (PID), I think it’s $\mathbb{Z}$. [...] But, of course, $\mathbb{Z}$ has nicer properties than the average PID. PIDs, for example, may not be Noetherian or have many other properties. So, you want to tell the students what a PID actually is. You say, ’okay, $\mathbb{Z}$ doesn’t quite cut it. I’ll give you a polynomial ring over a field as another example of a principal ideal domain, which is also a Euclidean domain.’ Maybe now they’re thinking PIDs are just places where you take the gcd. So, I give you an example of a PID which is not a Euclidean domain, and that’s meant to highlight the difference.” This, of course, comes back to providing learners access to a meaningful example space that would allow them to have a better mental image of what a structure is, and what it is not.

A richer example space consists of carefully selected dimensions of possible variation. Non-examples are things that clearly do not fit the definition, but share a surface level similarity. We could think of a set that might seem like it forms a group under a binary operation but it doesn’t. Boundary examples are those, which fit within a definition, but push its limits. Exercises to address this are possible, but they need to be carefully crafted.

2.2.3. *Inquiry-based Learning.* To sum up, in the early-knowledge acquisition phase, novices are not a blank slate and because of this and the evocative language used in definitions, misconceptions are likely to form and remain cemented. Later on, non-examples and boundary examples might be provided but it is essential that the initial forming of a concept image should be accurate, albeit not complete.

To counter this, educational researchers adopt Inquiry-based Learning (IBL) (for more on IBL implemented in abstract algebra, see eg. [23], [27], [12]). The core philosophy of IBL is that students engage in exploratory tasks prior to receiving formal definitions. Through exposure to examples, guided instruction, and opportunities to generate their own examples, students are encouraged to develop intuition and construct meaning independently.

Barton's summary captures the points succinctly and precisely: "Goldenberg and Mason (2008)[18] instead favour the use of examples before the presentation of any definition. They argue that students are more likely to attach meaning to the examples teachers go through, forming their own conceptions and conclusions, which can make up for the shortfalls in vocabulary and conceptual understanding of students that can render definitions alone pretty useless." [6, p. 223]

As is often the case, even in higher-education mathematics, when students are presented with a definition, one's first reaction might be "Where is this coming from?" or "Why is it defined this way?". Students might overlook the definition of something they consider useless or "too much to remember", and when faced with a technical task, they might only consult their incomplete concept image, and this is how misconceptions are formed and cemented. To put it simply, providing examples before giving the definition might help students regain this agency and ownership in their learning [48, p.30-31] . For example, an exploratory task as seen in our Appendix, will help learners find their "why" that is often so hidden and concealed in mathematics lectures.

The development of example spaces through learners generating their own examples was studied in depth by Watson and Mason (2006a [48], 2006b [49]). "Understanding is remembering in disguise" : schema growth is facilitated through the established iterative building on learner's previous knowledge. The authors argue that by constructing their own examples, students meaningfully engage with both concepts and procedures.

However, even the most carefully designed tasks and example spaces are insufficient if students are not given clear opportunities to check and refine their thinking. Without feedback in the early acquisition phase, misconceptions can remain hidden and uncorrected, and example spaces can inadvertently reinforce incorrect reasoning. In the next section, we identify the role that feedback plays in supporting students' conceptual development and addressing misconceptions before they become entrenched.

2.3. **Formative Feedback.** Formative feedback is one of the most powerful frameworks of tools that a teacher can implement but it is misunderstood. It's often met with remarks that teachers don't need more marking to do and that students lack the incentive to produce formative work because they prioritise summative assessments. True formative feedback is not just written comments or marked work that ends up ignored or forgotten by students (Wiliam, (2003), p, 43 [9]). Formative assessment (for more see Cowie and Bell, (1999) [13], Wiliam, (2011) [51]), involves techniques that help teachers and students understand learning progress and act on it in real time. This means that students must have the opportunity to respond to feedback and to make changes, and that teachers must be able to adapt their teaching based on what the feedback reveals. Authors also use terms like "Checking for Understanding" (Lemov, (2015) [28]) and "Assessment for Learning" (Wiliam et. al., (2003) [9]), each with its own specific definition and emphasis.

In many traditional lecture settings, however, this real-time feedback loop is often missing. Large class sizes and time constraints are frequently cited as reasons, but another key issue is uncertainty: many lecturers simply are not sure what actions they can take in a class of hundreds of students to make feedback actionable.

But there are practical strategies that work even in these environments. Two particularly powerful techniques that we will focus on are Concept-Tests (for more see Wiliam and Wylie, (2006) [54] or Feudel, Unger [17]) and peer and self-feedback (for more see Nicol, (2020) [35]).

2.3.1. *Concept-Tests.* In a traditional lecture setting, the lecturer often teaches content without actively checking whether students have really understood it (Lemov, (2015)). Imagine a student who mistakenly believes that the naive converse of Lagrange's theorem is true (which is, as we have seen, a common misconception). If the examples presented in the lecture happen to reinforce that belief, the student might carry it with full confidence all the way to the exam hall. Of course, it's not realistic nor efficient for the lecturer to personally question each student during a lecture, but if there are no opportunities throughout the year for either the teacher or the student to uncover such misconceptions, failure is almost inevitable. This is why techniques are needed: not to overload lecturers with more marking, but to systematically check for understanding as the learning is happening.

Concept Tests (or diagnostic questions) are one of the most effective ways of checking for understanding. Concept Tests are carefully designed multiple choice questions or true/false questions that reveal a specific misconception, oversimplification, or generalisation with each wrong option. By surfacing these misconceptions early, teachers can respond immediately and help to prevent students from building upon flawed foundations (Black , Wiliam (1998) [8] , Wiliam (2011)[51]). At the same time, these questions push students to reflect on their own thinking which is a way of self-feedback which we will explore in the next section. Additionally, because they are quick, low-stakes or no-stakes, and work with an entire class at once, Concept Tests are especially powerful in settings where individual checks of understanding aren't realistic.

Often the "answering" of such questions can happen in real-time through Classroom Response Systems or Audience Response Systems (eg. [3]). Lecturers at Manchester have embraced tools like Mentimeter to engage students during lectures. While it is a great platform for interaction, there is further potential to leverage it more for Concept Tests. One way an interviewee did gather this feedback at the end of class is by leaving such quizzes on the visualiser, leaving the students to think by themselves or share in pairs (Think-Pair-Share, see eg. Lyman (1981) [29]), and do a show of hands (could also be a Mentimeter vote). After the vote, they would explain why each wrong option was flawed. Also, depending on the vote, the lecturer would know where further emphasis was needed in the future to avoid these misunderstandings.

Concept Tests can be challenging to design. As one lecturer explained, "when you inherit a course you haven't taught before, it's hard to analyse and predict the misconceptions." This is precisely where our project—and our future work—comes in: to help design diagnostic questions tailored to specific courses. By building momentum, we also hope to encourage lecturers to make these resources publicly available. A sample set of these diagnostic questions can be found in the Appendix.

2.3.2. *The testing effect.* There is also a strong case to be made for formative tests and the testing effect. As Wiliam (2011) [51] highlights, every assessment can function formatively if it acts as a tool for learning. The testing effect (or retrieval effect) builds on research-based findings from distributed practice and Cognitive Load Theory. Distributed Practice (or the spacing effect), simply put, shows that long-term learning is more likely to occur when students revisit information at different points in time. The more frequently you try to retrieve information from your working memory, the stronger the pathway becomes. Think of it like finding a path through a forest you have not walked on in a while;f your old footprints help guide you, and each time you walk it, the path becomes clearer. This links back to our earlier definition of learning, "understanding is remembering in disguise", because once you are taught something, it becomes existing knowledge that must be actively retrieved to be cemented in long-term memory.

Interviewees also agreed that abstract algebra is highly cumulative: it builds on earlier concepts to reach higher-order ideas. If the foundational concepts are forgotten or misunderstood, further learning becomes unstable. The implication for teaching is that prior knowledge must be regularly reactivated. This could mean providing regular diagnostic quizzes, such as Concept Tests, on older content to reinforce these foundations. For more on this, see Roediger et. al. (2011) [40], Bjork and Bjork (2014) [7]. As established, formative quizzes don't have to take additional time away from lecturers; they can be integrated seamlessly into existing teaching. They could also be added as questions in tutorials or on exercise sheets. However, the most sustainable way to strengthen formative assessment with minimal lecturer intervention is by guiding students to practise doing peer-feedback and self-feedback.

2.3.3. *Peer-feedback.* Comparative judgement plays a major role in learning: internally comparing new knowledge or examples with pre-existing knowledge helps students to adjust and refine their concept images. It is how the process of abstraction we explained in section 1.1 works - "to abstract is to recognise what's shared across different experiences" - also means to recognise what is different.

It's the philosophy behind IBL and constructing example spaces. Therefore, it would be a shame not to recognise its power to leverage learning even further. One way to intentionally build on this natural process is through structured peer-feedback and self-feedback.

We will begin this section by playing the devil's advocate, and say that students don't know how to mark. While we do agree that when learners are in the early knowledge acquisition phase, you do not want to expose them to flawed work and risk that they internalise it as part of their concept images. But students don't actually have to mark anything in order to give feedback to each other. It is important to remember that the purpose of peer feedback is for students to clarify for one another what was correct or incorrect about their responses. Students should not give each other grades or scores. Rather, they should serve as reviewers who help to determine what might be lacking in their performance. Small groups are ideal for this, especially for exploratory work done in tutorials (see Appendix). For more, see eg. Jones and Alcock (2014) [25]. Conceptual change is socially mediated —the shift from misconception to accurate conceptualisation is much more likely to occur in the presence of others [47]. This is why verbal techniques like Think-Pair-Share can be so powerful: when students explain to each other their strategies for tackling a problem, they are more likely to uncover hidden misconceptions and act on them. One interviewee reported that while groups discuss a problem, they can intervene and correct any inaccurate beliefs. This highlights the fact that guided instruction remains important. For this reason, the prompts used in these discussions should be carefully chosen (see Appendix).

Most interviewees also observed an increasing trend in students using Piazza which is an online discussion forum where questions can be asked anonymously to each other and to their instructor. This highlights the benefits of new technologies and collaborating with one another.

2.3.4. *Self-feedback.* As part of a study skills course, first-year students had the chance to upload a proof, using Peer Scholar, an anonymous online peer learning platform. In groups of three, they anonymously reviewed each other's proofs by comparing them under a set of guiding questions. Then, after a few days (making use of the spacing effect), students returned to review their own work with fresh eyes.

This comparative framework can be amplified even further through a widely recognised technique: self-feedback. Following on from the thread of comparative judgement, and under pre-established guidance or training, students can begin to give themselves feedback on their own work. Self-feedback is believed to improve a novice's metacognition skills (thinking about their own thinking), enabling them to develop an expert-like understanding of wider contexts. In other words, novices are not yet self-regulated learners. Although they might be aware of the course's Intended Learning Outcomes, they often do not know how to situate themselves against these goals, nor do they know what exactly to improve upon.

Nicol (2021)[35] defined self-feedback, also called internal feedback, as such: "Internal feedback is the new knowledge that students generate when they compare their current knowledge and competence against some reference information." The key components in Nicol's model are some type of external information/comparator (this could be worked examples by the teacher, a peer's solution, a model solution or even a solution provided by AI) and internal construction. As students reflect on their ongoing work in relation to their goals and the strategies they have employed to reach them, they create internal feedback. This feedback is generated in real-time and students naturally act on it during learning.

The implications of these techniques are significant. In the Appendix, we have provided some guiding questions that could be integrated into take-home exercise sheets to help students to develop their self-feedback skills. These questions provide a form of structured training, prompting learners to reflect deeply on their own work. Examples include: How does this solution differ from yours? What did you learn from that difference? (adapted from Nicol and McCallum, 2022 [36]). Another example is: Have I considered all necessary cases or conditions in my solution, and do these appear in the comparator? (see the Appendix for further questions).

An interviewee gave a notable insight, unaware of the existing research supporting it: "I always tell my students to highlight where in the proof they got stuck." By pinpointing the exact moment their understanding breaks down, students generate internal feedback, and they are prompted to reflect on why their understanding broke down. Over time, this habit of pausing and reflecting and fixing gaps in reasoning is fruitful in building self-regulated learners. Another technique that is also grounded in self-feedback and was shared by a different interviewee, involves designing lecture notes with colour-coded examples and non-examinable content. Similarly, in the Level 2 Groups and Geometry course, the exercise sheets clearly indicate the difficulty level of each question in the header. These strategies give students a better sense of their own place within the educational progression, and help to improve their self-regulation skills.

3. Proof Comprehension

The past subsections have hinted at a greater need for feedback, particularly in proof comprehension and writing. Abstract algebra, rich with proofs, relies on them as the essential bridge from axioms to theorems. While our focus so far has been on techniques to improve conceptual understanding, proofs uniquely combine this with procedural fluency, making them a critical skill for learners to master.

Most interviewees reported noticing struggles amongst learners with proof comprehension. "They don't know which assumption is used in which part of the proof.", "They don't understand why I am doing the next step and I have to explain again." Research supports these observations (eg. Alcock and Weber, (2005) [2] and Weber, (2008) [50]). In the following subsections, we will try to scratch the surface of the established techniques that can help learners with proof comprehension.

3.1. **Proof Summaries.** In this part, we build on the work of Davies and Jones (2022) [14], who explore assessing proof comprehension through the use of proof summaries. Their research draws on earlier studies by Raman (2003)[39], Yan (2019)[55], Mejia-Ramos (2017) [33] and Inglis (2009)[32], among others.

A proof summary can be understood as a carefully crafted, concise representation of a formal proof that maintains the necessary level of rigour and detail to accurately reflect the original argument. Firstly, this task requires learners to understand the original proof thoroughly, through techniques such as self-explanation. In order to create a summary, they need to abstract the technical details from the essential argument. Coming back to section 1.3, novices are known to focus on the superficial particularities of the problem they are facing. An interviewee also agrees, "First-year students tend to be so concerned with the calculation in the proof, instead of the overarching argument." Therefore, proof summaries serve as an excellent tool to foster the expert-like ability to identify what is truly important in a proof and to recognize how such arguments can be applied in different contexts. Secondly, Davies argues, the summarising task becomes a hybrid activity, involving both comprehension and construction.

Having the principles of distributed practice, peer and self-feedback in mind, there are plenty of ways that proofs summaries can be implemented in lectures or tutorials.

One way to implement them is by setting aside time at the end of a lecture once a section is complete for students to write a short synopsis of a significant proof that was covered during that session. The proof can be indicated by the lecturer and this quick exercise can be repeated in tutorials. Through distributed practice, students can later return to their proof summaries when revising, effectively explaining the material to themselves and reinforcing their understanding. Proof summaries could also be implemented as part of the tutorial sheet as an individual task. This could then be paired with group work, where students in small groups review each other's proof summaries and, at the end, return to update their own. Ultimately, keeping a log of proof summaries could aid in creating a cheat sheet for partially open-book exams. An interviewee added, "students tell me that by them working on the cheat sheet it's almost as if they already studied and they don't need it anymore." This aligns with the theory behind proof summaries, as the act of summarising requires deep processing and active engagement with the proof.

We have been alluding to self-explanation throughout this section, and now it is time to highlight one of the most powerful and amazing techniques in mathematics education: self-explanation training.

3.2. **Self-explanation.** Knowing what we know now about comparative judgement and self-feedback, it should feel intuitive to introduce self-explanation. Chi et. al. (2000) conducted significant studies on this, defining self-explanation as follows: "the activity of generating explanations to oneself [...] a knowledge-building activity that is generated by and directed to oneself." [10]

The effectiveness of self-explanation training in mathematics, with particular focus on proof comprehension is explored by Hodds, Alcock and Inglis in [22]) .

They outlined its key principles as identifying the main ideas in each line of a proof and explaining each line by connecting it either to earlier steps within the proof or to prior knowledge.

Providing a parallel with the self-feedback section, novices are not yet self-regulated learners. They don't naturally ask themselves the guiding questions that are required to engage with every step of the proof and to critically analyse it. However, we know that they are capable of comparing their own thinking, strategies, and prior knowledge with the proof in front of them. Therefore, to unlock the true learning potential of self-explanation, students must be explicitly taught how to self-explain. (see Mejia Ramos (2017) [33]). For instance, students might be prompted to pause after each line and identify the main idea being expressed. Then, they might be asked to explain the aforementioned idea in relation to earlier steps in the proof, previous theorems, or their own prior knowledge of the topic. "Do I fully understand the idea used in this line?", "Why was this idea or method chosen here?", "How does this step connect to the overall proof, other theorems, or concepts I already know?.

In practice, self-explanation is, too, a seamless way to implement a powerful teaching tool in university courses. A self-explanation prompt could be given in take-home exercise sheets, an example on (the fundamental isomorphism theorem is provided in the Appendix. Alternatively, if time permits during lectures, students could be given a short pause after a worked example from the instructor to self-explain the reasoning behind each step and to write down their explanations down as they go. Self-explanation feels like a natural and fitting culmination of this project. Not only because it has been shown to be highly effective, but also because we have built the right foundational frameworks throughout to fully understand why it works so well.

4. Conclusion

The key takeaway for readers is the fundamental nature of learning: we continuously learn by comparing new material with what we already know, and by actively retrieving that knowledge over time. Learning mathematics is no different. Some parts of it happen to be especially more complex because they are abstract in nature.

Together, all the above strategies highlight how deliberate reflection can transform students' engagement with proofs and abstract concepts, while taking ownership of their learning. Ultimately, these emphasise the intentionality that students so eagerly crave for in maths lectures.

Mathematics is drawing on the human ability of imagination, and abstract algebra is often misunderstood as rigid when a lot of creativity lies beneath it. This project is both an expression of appreciation for abstract algebra and, as a student, a small attempt to contribute to teaching it in a way that honours its beauty and depth.

Looking ahead, these strategies could be explored more widely across different areas of mathematics education or refined to suit diverse learners. A repository of Concept Tests, active self-feedback and self-explanation prompts could be created following the Appendix. These techniques could be implemented in abstract algebra course units and their efficacy could be observed in the future.

References

[1] L. Alcock and A. Simpson. *Ideas from mathematics education : an introduction for mathematicians*. Higher Education Academy, 2009.

[2] L. J. Alcock and K. Weber. "Proof validation in real analysis: Inferring and checking warrants". In: *Journal of Mathematical Behavior* 24 (2005), pp. 125–134. DOI: 10.1016/j.jmathb.2005.03.003.

[3] J. V. Alviar and A. E. Gamorez. "Effects of Classroom Response System on the Achievement and Knowledge Retention of the Students in Mathematics". In: *Cogent Education* 11.1 (2024). DOI: 10.1080/2331186X.2024.2323364. URL: https://doi.org/10.1080/2331186X.2024.2323364.

[4] P. E. Andrew J. Martin. "Load reduction instruction: Exploring a framework that assesses explicit instruction through to independent learning". In: *Teaching and Teacher Education* 73 (2018), pp. 203–214. DOI: https://doi.org/10.1016/j.tate.2018.03.018.

[5] M. Asiala et al. "Development of students' understanding of cosets, normality, and quotient groups". In: *The Journal of Mathematical Behavior* 16.3 (1997). ISSN: 0732-3123. DOI: https://doi.org/10.1016/S0732-3123(97)90029-8.

[6] C. Barton. *How I Wish I Had Taught Maths: Reflections on research, conversations with experts, and 12 years of mistakes*. John Catt Educational Ltd, 2018.

[7] E. L. Bjork and R. A. Bjork. "Making Things Hard on Yourself, but in a Good Way: Creating Desirable Difficulties to Enhance Learning". In: *Psychology and the Real World: Essays Illustrating Fundamental Contributions to Society*. 2nd. New York, NY: Worth Publishers, 2014, pp. 59–68.

[8] P. Black and D. Wiliam. "Inside the Black Box: Raising Standards Through Classroom Assessment". In: *Phi Delta Kappan* 80.2 (1998), pp. 139–148.

[9] P. Black et al. *Assessment for Learning: Putting It into Practice*. Buckingham: Open University Press, 2003.

[10] M. T. H. Chi. "Self-explaining: The Dual Processes of Generating Inferences and Repairing Mental Models". In: *Advances in Instructional Psychology: Educational Design and Cognitive Science*. Vol. 5. Lawrence Erlbaum Associates, 2000, pp. 161–238.

[11] M. T. H. Chi, P. J. Feltovich and R. Glaser. "Categorization and Representation of Physics Problems by Experts and Novices". In: *Cognitive Science* 5.2 (1981), pp. 121–152.

[12] J. P. Cook. "The emergence of algebraic structure: Students come to understand units and zero-divisors". In: *International Journal of Mathematical Education in Science and Technology* 45.3 (2014), pp. 349–359. DOI: 10.1080/0020739X.2013.837526. URL: https://doi.org/10.1080/0020739X.2013.837526.

[13] B. Cowie and B. Bell. "A model of formative assessment in science education". In: *Assessment in Education: Principles, Policy & Practice* 6.1 (1999), pp. 101–116.

[14] B. Davies and I. Jones. "Assessing Proof Reading Comprehension Using Summaries". In: *International Journal of Research in Undergraduate Mathematics Education* 8 (2022), pp. 469–489. DOI: 10.1007/s40753-021-00157-6. URL: https://doi.org/10.1007/s40753-021-00157-6.

[15] D. Didau. "A Novice→Expert Model of Learning". In: *The Learning Spy Blog* (2017).

[16] E. Dubinsky et al. "On learning fundamental concepts of group theory". In: *Educational Studies in Mathematics* 27.3 (1994).

[17] F. Feudel and A. Unger. "Development and Implementation of Concept-Test Questions in Abstract Algebra". In: *International Journal of Mathematical Education in Science and Technology* 56.5 (2024).

[18] P. Goldenberg and J. Mason. "Shedding light on and with example spaces". In: *Educational Studies in Mathematics* 69.2 (2008).

[19] O. Hazzan. "Reducing abstraction level when lcarning abstract algebra concepts". In: *Educational Studies in Mathematics* 40.1 (1999). DOI: https://doi.org/10.1023/A:1003780613628.

[20] O. Hazzan and U. Leron. "Students' use and misuse of mathematical theorems: The case of Lagrange's theorem". In: *For the Learning of Mathematics* 16.1 (1996).

[21] N. M. Hill and W. Schneider. "Brain changes in the development of expertise: neuroanatomical and neurophysiological evidence about skill-based adaptations". In: *The Cambridge handbook of expertise and expert performance* (2006).

[22] M. Hodds, L. Alcock and M. Inglis. "Self-explanation training improves proof comprehension". In: *Journal for Research in Mathematics Education* 45 (2014), pp. 62–101. DOI: 10.1080/03075079.2013.821974. URL: https://doi.org/10.1080/03075079.2013.821974.

[23] J. Hodge et al. *Abstract Algebra: An Inquiry-Based Approach.* CRC Press, 2023.

[24] M. Inglis and L. Alcock. "Expert and novice approaches to reading mathematical proofs". In: *Journal for Research in Mathematics Education* 43 (2012), pp. 358–390. DOI: https://doi.org/10.5951/jresematheduc.43.4.0358.

[25] I. Jones and L. Alcock. "Peer assessment without assessment criteria". In: *Studies in Higher Education* 39 (2014), pp. 1774–1787. DOI: 10.1080/03075079.2013.821974. URL: https://doi.org/10.1080/03075079.2013.821974.

[26] P. A. Kirschner, J. Sweller and R. E. Clark. "Why minimal guidance during instruction does not work: an analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching". In: *Educational Psychologist* 41.2 (2006).

[27] S. Larsen and E. Lockwood. "A local instructional theory for the guided reinvention of the quotient group concept". In: *The Journal of Mathematical Behavior* 32.4 (2013), pp. 726–742. ISSN: 0732-3123. DOI: 10.1016/j.

jmathb.2013.02.010. URL: https://doi.org/10.1016/j.jmathb.2013.02.010.

[28] D. Lemov. *Teach Like a Champion 2.0: 62 Techniques That Put Students on the Path to College*. John Wiley & Sons, 2015.

[29] F. Lyman. "The Responsive Classroom Discussion". In: *Mainstreaming Digest*. Ed. by A. S. Anderson. College Park, MD: University of Maryland College of Education, 1981, pp. 109–113.

[30] F. Marton and M. F. Pang. "Meanings are acquired from experiencing differences against a background of sameness, rather than from experiencing sameness against a background of difference: putting a conjecture to the test by embedding it in a pedagogical tool". In: *Frontline Learning Research* 1.1 (2013).

[31] P. McCrea. *Memorable Teaching: Leveraging Memory to Build Deep and Durable Learning in the Classroom*. CreateSpace Independent Publishing Platform, 2017.

[32] J. P. Mejía-Ramos and M. Inglis. "Argumentative and Proving Activities in Mathematics Education Research". In: *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education*. Vol. 2. 2009, pp. 88–93.

[33] J. P. Mejía-Ramos et al. "Developing and Validating Proof Comprehension Tests in Undergraduate Mathematics". In: *Research in Mathematics Education* 19.2 (2017), pp. 130–146. DOI: 10.1080/14794802.2017.1325776. URL: https://doi.org/10.1080/14794802.2017.1325776.

[34] K. Melhuish and M. D. Ellis B. Hicks. "Group theory students perceptions of binary operation". In: *Educational Studies in Mathematics* 103.1 (2020). DOI: https://doi.org/10.1007/s10649-019-09925-3.

[35] D. Nicol. "The power of internal feedback: exploiting natural comparisons". In: *Assessment & Evaluation in Higher Education* 46.3 (2021), pp. 435–450.

[36] D. Nicol and S. McCallum. "Making internal feedback explicit: exploiting the multiple comparisons that occur during peer review". In: *Assessment & Evaluation in Higher Education* 47.3 (2022), pp. 424–443.

[37] J. Piaget. *The development of thought : equilibration of cognitive structures*. Viking Press, 1977.

[38] J. Piaget. *Judgment and Reasoning in the Child*. New York: Harcourt, 1928.

[39] M. Raman. "Key Ideas: What Are They and How Can They Help Us Understand How People View Proof?" In: *Educational Studies in Mathematics* 52.3 (2003), pp. 319–325. DOI: 10.1023/A:1024360304324. URL: https://doi.org/10.1023/A:1024360304324.

[40] H. L. Roediger, A. L. Putnam and M. A. Smith. "Ten Benefits of Testing and Their Applications to Educational Practice". In: *Psychology of Learning and Motivation: Cognition in Education*. Ed. by J. P. Mestre and B. H. Ross. Vol. 55. San Diego, CA: Elsevier Academic Press, 2011, pp. 1–36.

[41] A. Selden and J. Selden. "Validations of Proofs Considered as Texts: Can Undergraduates Tell Whether an Argument Proves a Theorem?" In: *Journal for Research in Mathematics Education* 34.1 (2003), pp. 4–36. DOI: https://doi.org/10.2307/30034698.

[42] R. Skemp. “Relational Understanding and Instrumental Understanding”. In: *The Arithmetic Teacher* 26.3 (1976). DOI: https://doi.org/10.5951/at.26.3.0009.

[43] R. Skemp. *The Psychology of Learning Mathmatics*. Erlbaum, 1971.

[44] J. Sweller. “Cognitive load during problem solving : effects on learning”. In: *Cognitive Science* (1988). DOI: https://doi.org/10.1016/0364-0213(88)90023-7.

[45] S. Tall D. O. Vinner. “Concept image and concept definition in mathematics with particular reference to limits and continuity”. In: *Educational Studies in Mathematics* 12 (1981), pp. 151–163.

[46] S. Vinner. “The role of definitions in teaching and learning”. In: *Advanced Mathematical Thinking* (1991), pp. 65–81.

[47] L. S. Vygotsky. *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press, 1978.

[48] A. Watson and J. Mason. *Mathematics as a Constructive Activity*. Routledge, 2006.

[49] A. Watson and J. Mason. “Variation and mathematical structure”. In: *Mathematics Teaching incorporating Micromath* (2006).

[50] K. Weber. “How Mathematicians Determine If an Argument Is a Valid Proof”. In: *Journal for Research in Mathematics Education* 39.4 (2008), pp. 431–459.

[51] D. Wiliam. *Embedded Formative Assessment*. Solution Tree Press, 2011.

[52] D. T. Willingham. *Why don’t students like school? : a cognitive scientist answers questions about how the mind works and what it means for the classroom*. John Wiley Sons, 2009.

[53] P. S. Wilson. “Feature frequency and the use of negative instances in a geometric task”. In: *Journal for Research in Mathematics Education* 17.2 (1986).

[54] E. C. Wylie and D. Wiliam. “Diagnostic Questions: Is There Value in Just One?” In: *Annual Meeting of the National Council on Measurement in Education*. San Francisco, CA, 2006.

[55] X. Yan. “Key Ideas in a Proof: The Case of the Irrationality of $\sqrt{2}$”. In: *Journal of Mathematical Behavior* 55 (2019), pp. 1–10. DOI: 10.1016/j.jmathb.2019.04.001. URL: https://doi.org/10.1016/j.jmathb.2019.04.001.

Appendix - Abstract Algebra Activities

Below are some suggested activities for abstract algebra course units that apply the research findings and theoretical frameworks described in the report.

(1) **Inquiry based learning**
Using inquiry based learning students explore properties and patterns of algebraic structures using standard examples. This develops conceptual understanding and an appreciation of the definitions and theorems in the course. Ideally the students should solve the problems in small groups so they can discuss their ideas. Below are a couple of examples based on the Teaching Abstract Algebra for Understanding project (Larsen and Lockwood (2013),[27]).

To help students understand cosets:

(i) Choose a small finite group G eg. D_8, S_3 or A_4 and write down all the subgroups of G.
(ii) Let H be a proper subgroup of G. Write down all the left cosets of H. What do you notice about the cosets? Can you find a rule for when two cosets gH and hH are equal?
(iii) Do the same thing with the right cosets of H. Is H a normal subgroup of G?

To help students understand the properties of a factor group:
Let $(G, \star)$ be a finite group and let $K = \{A_1, A_2, ...A_k\}$ be a partition of G (ie. the subsets $A_i, i = 1, ..., k$ are pairwise disjoint and their union is G).
Assume that K is a group with operation $\times$ given by

$$A_i \times A_j = \{a_i \star a_j | a_i \in A_i, a_j \in A_j\},$$

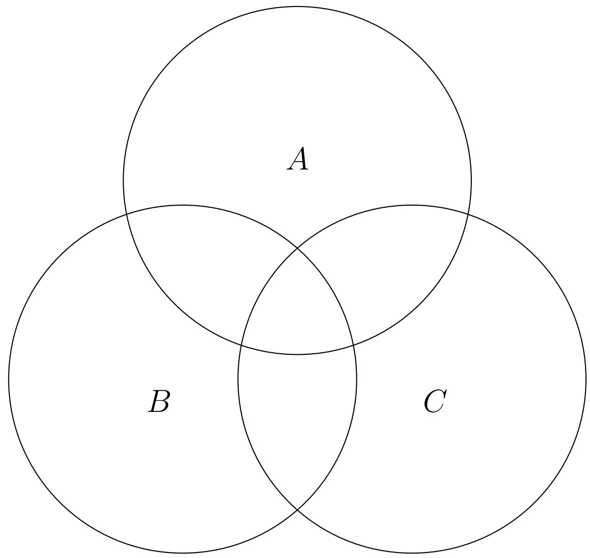
for all $i, j = 1, ..., k$.

(i) Show that the identity element of K, is a subgroup of G. Call this subgroup H.
(ii) Show that the elements of K are all the left cosets of H.
(iii) Show that $gH = Hg$ for all $g \in G$ ie. the corresponding left and right cosets of H are equal and H must be a normal subgroup of G.

If it is too difficult to do this for a general group G, get them to do this with a specific small group.
For more inquiry based learning activities in abstract algebra see Hodge, Schlicker and Sundstrom (2023) [23].

(2) **Finding examples - see Watson and Mason (2006) [48]**
This activity can be done as a group. Finding particular examples helps students recall key definitions and develops fluency and concept images.
The Venn diagram below represents the set of all rings. Let A be the set of all commutative rings, B be the set of all division rings and C be the

set of all domains. Find examples of rings that go in each region of the diagram or say why there are no examples for a particular region.



(3) **Active self-feedback using comparative judgement - see Nicol, (2021) [35]**

(i) Students write out a solution to an exercise.
(ii) They compare their solution to a comparator. This could be another student's solution, a model solution or a solution created by AI etc.
(iii) Students write down answers to some questions, for example:
  - Is my solution as logically sound and rigorous as the comparator? If not, what are the differences?
  - Are there any steps in my solution or the comparator that rely on unstated assumptions? If so what are they?
  - How does the clarity and precision of my mathematical language compare to the comparator?
  - Am I using definitions and theorems correctly and explicitly?
  - Is my solution as concise and elegant as the comparator; or are there redundant steps or unnecessarily complex arguments?
  - Have I considered all necessary cases or conditions in my solution and do these appear in the comparator?
(iv) Students should then make improvements to their solution.

This activity could be adapted to also use peer feedback if the comparator is another student's work. This may cause embarrassment if done face to face so an online tool such as peerScholar (https://www.peerscholar.com) could be used to make this process anonymous.

(4) **Proof comprehension**

Use **self-explanation training** on a proof of a theorem from the lecture notes (see Hodds, Alcock and Inglis (2014) [22] and Mejia Ramos et al (2017) [33]).
Write out each step in the proof on a separate line. For each step:

- What concepts are used in the step? Explain how these concepts have been used.
- What previous information is being used? This may be assumptions in the theorem, definitions, previous steps in the proof, previous theorems in the course.

For the whole proof:

- Describe the method of proof used and the overall logical structure of the proof.
- What are the main ideas in the proof?
- Where are the conditions and assumptions in the statement of the theorem used in the proof?
- Write down any steps in the proof that you don't understand.

Example of step-by-step explanations: The Fundamental Isomorphism Theorem (from Rings and Fields)

Let $\theta : R \to S$ be a ring homomorphism. Then $R/ker(\theta) \simeq im(\theta)$.

Proof - self-explanation in bold.

(i) Let $I = ker(\theta)$.
I is an ideal. We have already proved that $ker(\theta)$ is a proper ideal of R.

(ii) Let $\theta' : R/I \to S$ be defined by $\theta'(r + I) = \theta(r)$ for all $r \in R$.
θ' is defined on the factor ring R/I and so the elements in the domain are cosets $r+I$. We have already shown that θ' is a well-defined function in Lemma 4.4, ie. if $r+I = s+I$, then $\theta'(r + I) = \theta'(s + I)$. We need to check this because a coset can be represented by any element of the coset.

(iii) Then θ' is a ring homomorphism.
This was shown in Lemma 4.4. We used the definitions of addition and multiplication of cosets and the fact that θ is a homomorphism.

(iv) We have $\theta'(r+I) = 0$ iff $\theta(r) = 0$ iff $r \in ker(\theta) = I$ iff $r+I = 0+I$ and so $ker(\theta') = \{0 + I\}$.
There is a lot going on in this step! First we use the definition of θ' to show that $\theta'(r + I) = 0$ iff $\theta(r) = 0$. The second iff is using the definition of the kernel and the fact we have defined $I = ker(\theta)$. The third iff uses the property of cosets that $r + I = s + I$ iff $r - s \in I$ in the special case $s = 0$. This all shows that $\theta'(r + I) = 0$ iff $r + I = 0 + I$. This means that the only element in the kernel of θ' is the coset $0 + I$, the zero in R/I.

(v) By Lemma 3.15, θ' is injective.
This lemma states that a homomorphism is injective iff the kernal only contains the zero element. This is what we showed for θ' in the previous step.
(vi) Therefore $R/ker(\theta) \simeq im(\theta') = im(\theta)$.
In the previous step we showed that θ' is an injective homomorphism. We then use the property of an injective function (proved in a previous course) that there is a bijection between the domain and the image of the function. In this case that means that $R/I \simeq im(\theta')$. Finally we use the definition of θ' to show that $im(\theta') = im(\theta)$.

(5) **Proof summaries - see Davies and Jones (2022) [14]**
In fewer than 50 words summarise a proof from the notes. This can be used alongside self-explanation training. For the example of the Fundamental Isomorphism Theorem a summary could be:

This theorem is important and is used to understand the structure of factor rings. The proof defines another function $\theta' : R/ker(\theta) \to S$ which is shown to be an injective homomorphism, using previous results in the course. Properties of injective functions then give the required isomorphism.

(6) **Concept Tests** (see Feudal and Unger (2024)[17], and Wiliam and Wylie, (2006) [54])
A series of diagnostic quizzes that can be done in class or online. Ideally these should challenge students to recall definitions and theorems and link different parts of the course and make use of common misconceptions (some common misconceptions and concept tests for these for abstract algebra are given in Feudal and Unger).
MCQs, Multiple answer or True/False questions work well here but time should be spent on explaining the answers, to explain any misconceptions. For example a common misconception in group theory is to confuse normality with commutativity ie. a subgroup H is normal in G iff $gh = hg$ for all $g \in G$ and for all $h \in H$. Think Pair Share could also be used here to encourage peer learning:
(i) Students answer the questions individually.
(ii) They discuss their answers in small groups.
(iii) They agree on a common answer.
(iv) Answers are then discussed with the whole group.

Examples of Concept Tests from Rings and Fields. True or False? Explain your answers.
- If $(R, +, \times)$ is a ring, then $(R\backslash\{0\}, \times)$ is a group.
- Every integral domain is a field.
- Every non-zero nilpotent element of a ring is a zero divisor.

- The ring of polynomials over a field is a field.
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \mathbb{Z}_4$.

  Let $\theta : R \to S$ be a ring homomorphism.
- If R is commutative, then S is commutative.
- The image of θ is a ring.
- If $\theta(r)$ is a unit in S, then r is a unit in R.
- $ker(\theta)$ is an ideal of R.

  Let I be a proper ideal of a ring R.
- $(I, +)$ is a subgroup of $(R, +)$.
- If I is maximal, then it contains all the proper ideals of R
- I is a principal ideal.
- The ideals of R/I are also ideals of R.
- If R is commutative and I is maximal, then R/I is an integral domain.

  Let K be any field and $K[X]$ be the ring of polynomials over K.
- If $f, g \in K[X]$, then $< f, g >$ is a principal ideal.
- If $f \in K[X]$ is irreducible, then f has no roots in K.
- $X^2 + 1$ is irreducible in $K[X]$.
- $K[X]/ < X^2 + X >$ is a field.
- $K[X]/ < X > \simeq K$.

(7) **Distributed learning - see Wiliam (2011) [51]**
At the end of each section of the notes get students to write a short summary of the main concepts in the section and a list of things they need to review later. These summaries can be used during revision and to create a cheat sheet for partially open book tests and exams.